ON THE STABILITY OF THE FLOW OF A VISCOELASTIC FLUID DOWN AN INCLINED PLANE

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ABSTRACT: The paper presents an analysis of laminar flow of a film of viscoelastic fluid flowing under gravity down an infinite inclined plane. It is assumed that the mechanical behavior of the fluid can be represented by a generalized Maxwell model, whose constitutive equation contains a time derivative of the deviator of the stress tensor in the Jaumann sense [1, 2]. The equations of motion of the viscoelastic fluid considered here admit an exact solution for the case of rectilinear laminar flow with a plane free boundary. The stability of this flow with respect to surface waves is investigated by the method of successive approximations described in [3, 4].

1. The viscoelastic flow of several real fluids can be approximately described by the generalized Maxwell model [2]

$$\varepsilon_{ii} = 0, \quad s_{ik} = 2\mu\varepsilon_{ik} - \lambda\left(\frac{\partial s_{ik}}{\partial t^*} + v_j\frac{\partial s_{ik}}{\partial x_j} + \omega_{ij}s_{jk} + \omega_{kj}s_{ij}\right) \quad (1.1)$$

$$\varepsilon_{ik} = \frac{1}{2}\left(\frac{\partial v_k}{\partial x_i} + \frac{\partial v_i}{\partial x_k}\right), \qquad \omega_{ik} = \frac{1}{2}\left(\frac{\partial v_k}{\partial x_i} - \frac{\partial v_i}{\partial x_k}\right) \quad (i, j, k = 1, 2, 3).$$

Here s_{ik} are the components of the deviator of the stress tensor, v_j are the components of the velocity vector, μ is the coefficient of viscosity, λ is the relaxation time, and x_j are rectangular Cartesian coordinates.

The viscoelastic flow of a relaxing fluid with finite rates of strain is governed by (1.1) and the equations of motion

$$\rho\left(\frac{\partial v_k}{\partial t^*} + \frac{\partial v_k}{\partial x_j}v_j\right) = -\frac{\partial p^*}{\partial x_k} + \frac{\partial s_{kj}}{\partial x_j} + F_k \quad (j, k = 1, 2, 3). \quad (1.2)$$

Here F_k are the projections of the body force, ρ is the density, and p^* is the hydrostatic pressure. To analyze the stability of the flow of a film of thickness d flowing under the action of gravity down a plane inclined at an angle β with respect to the horizontal, we introduce the dimensionless variables

$$t = \frac{t^*k}{d}, \quad p = \frac{p^*}{\rho k^2}, \quad x = \frac{x_1}{d}, \quad u = \frac{v_1}{k}, \quad v = \frac{v_2}{k}$$

$$y = \frac{x_2}{d}, \quad s_{xx} = \frac{s_{11}}{\rho k^2}, \quad s_{yy} = \frac{s_{22}}{\rho k^2}, \quad s_{xy} = \frac{s_{12}}{\rho k^2}, \quad k = \frac{\rho g d^2 \sin\beta}{3\mu},$$

where g is the acceleration of gravity.

In the case of two-dimensional unsteady flow with $F_1 = \rho g \sin\beta$, $F_2 = \rho g \cos\beta$, $F_3 = 0$, Eqs. (1.1), (1.2) take the form

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}u + \frac{\partial u}{\partial y}v = -\frac{\partial p}{\partial x} + \frac{3}{R} + \frac{\partial s_{xx}}{\partial x} + \frac{\partial s_{xy}}{\partial y},$$

$$\frac{\partial v}{\partial t} + \frac{\partial v}{\partial x}u + \frac{\partial v}{\partial y}v = -\frac{\partial p}{\partial y} + \frac{3\,\mathrm{ctg}\,\beta}{R} + \frac{\partial s_{xy}}{\partial x} + \frac{\partial s_{xx}}{\partial y},$$

$$s_{xx} = -s_{yy} = \frac{2}{R}\frac{\partial u}{\partial x} - \tau\left[\frac{\partial s_{xx}}{\partial t} + \frac{\partial s_{xx}}{\partial x}u + \frac{\partial s_{xx}}{\partial y}v + s_{xy}\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial y}\right)\right],$$

$$s_{xy} = \frac{1}{R}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) - \tau\left[\frac{\partial s_{xy}}{\partial t} + \frac{\partial s_{xy}}{\partial x}u + \frac{\partial s_{xy}}{\partial y}v + s_{xx}\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial x}\right)\right],$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad \left(R = \frac{kd\rho}{\mu}, \ \tau = \frac{\lambda k}{d}\right). \quad (1.3)$$

These equations are satisfied by the relations

$$s_{xy}^\circ = -\frac{3}{R}y, \qquad s_{xx}^\circ = -\frac{3\tau}{R}\frac{du^\circ}{dy}, \qquad p^\circ = -s_{xx}^\circ + \frac{3\,\mathrm{ctg}\,\beta}{R}y,$$

$$u^\circ = \frac{1}{6\tau^2}\ln\frac{1 + \sqrt{1 - 36\tau^2}}{1 + \sqrt{1 - 36\tau^2 y^2}} + \frac{1}{6\tau^2}\left(\sqrt{1 - 36\tau^2 y^2} - \sqrt{1 - 36\tau^2}\right),$$

$$\frac{du^\circ}{dy} = \frac{-1 + \sqrt{1 - 36\tau^2 y^2}}{6\tau^2 y}, \quad (1.4)$$

which describe the steady flow of a film with a plane stress-free surface, flowing under gravity down an inclined plane, with the no-slip condition at the plane y = 1 (Fig. 1).

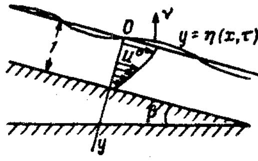

Fig. 1

From (1.4) it can be seen that rectilinear flow of the film is possible when $\tau < 1/6$. When $\tau = 0$, equations (1.4) describe the flow of a film of Newtonian fluid [4].

We expand the velocity $u^\circ(y, \tau)$ in a power series of the small parameter τ, and truncate the expansion of each expression in (1.4) after the τ^2 term. The result is

$$s_{xy}^\circ = -\frac{3}{R}y, \qquad s_{xx}^\circ = -s_{yy}^\circ = \frac{9\tau}{R}y^2,$$

$$u^\circ = \frac{3}{2}(1 - y^2) + \frac{27}{4}\tau^2(1 - y^4),$$

$$\frac{du^\circ}{dy} = -3y - 27\tau^2 y^3, \quad (1.5)$$

$$p^\circ = -\frac{9\tau}{R}y^2 + \frac{3\,\mathrm{ctg}\,\beta}{R}y.$$

2. We proceed to investigate the stability of solution (1.5) with respect to two-dimensional perturbations. Consider a two-dimensional unsteady flow of the form

$$u = u^\circ + u'(t, x, y), \quad v = v'(t, x, y), \quad p = p^\circ + p'(t, x, y),$$

$$s_{xx} = s_{xx}^\circ + s_{xx}'(t, x, y), \quad s_{xy} = s_{xy}^\circ + s_{xy}'(t, x, y). \quad (2.1)$$

Here u°, p°, s_{xx}°, s_{xy}° are parameters of the basic flow (1.5), and u', s_{xx}', s_{xy}', v', p' are the two-dimensional perturbations.

At the perturbed free surface $y = \eta(t, x)$ we have the kinematic and dynamic conditions [4]

$$v = \frac{\partial\eta}{\partial t} + u\frac{\partial\eta}{\partial x},$$

$$s_{xx}[\cos^2(\nu, x) - \cos^2(\nu, y)] + 2s_{xy}\cos(\nu, x)\cos(\nu, y) - p = -S\frac{\partial^2\eta}{\partial x^2},$$

$$2s_{xx}\cos(\nu, x)\cos(\nu, y) + s_{xy}[\cos^2(\nu, y) - \cos^2(\nu, x)] = 0,$$

$$(S = T/\rho d k^2). \quad (2.2)$$

Here ν is the outer normal to the free surface, and T is the coefficient of surface tension.

At the surface y = 1 we have the conditions $u'(t, x, 1) = 0$, $v'(t, x, 1) = 0$. We linearize (1.1), (2.2), taking account of (1.5) and (2.1). Introducing the perturbation stream function by means of the definitions $u' = \partial\Psi/\partial y$, $v' = -\partial\Psi/\partial x$, we represent the perturbations in the form

$$p' = f(y)e^{i\alpha(x-ct)}, \quad s_{xx}' = s_{xx}^*(y)e^{i\alpha(x-ct)},$$

$$s_{xy}' = s_{xy}^*(y)e^{i\alpha(x-ct)}, \quad \psi = \varphi(y)e^{i\alpha(x-ct)}.$$

The variables $f(y)$, $\varphi(y)$, $s_{xx}^*(y)$, $s_{xy}^*(y)$ are determined by the system

$$i\alpha(u^\circ - c)\frac{d\varphi}{dy} - i\alpha\frac{du^\circ}{dy}\varphi = -i\alpha f + i\alpha s_{xx}^* + \frac{ds_{xy}^*}{dy},$$

$$\alpha^2\varphi(u^\circ - c) = -\frac{df}{dy} + i\alpha s_{xy}^* - \frac{ds_{xx}^*}{dy},$$

$$Rs_{xy}^* = \theta_1\left(\frac{d^2\varphi}{dy^2} + \alpha^2\varphi\right) + 18\tau^2 y^2\left(\alpha^2\varphi - \frac{d^2\varphi}{dy^2}\right);$$

$$+ \theta_2\left(6i\alpha y \frac{d\varphi}{dy} - 3i\alpha\varphi\right) + 6\tau^2\alpha^2 y\,(u^\circ - c),$$

$$Rs_{xx}{}^* = 2\theta_1 i\alpha \frac{d\varphi}{dy} - 6\tau\, y \frac{d^2\varphi}{dy^2} +$$

$$+ 9y\tau^2 i\alpha\,(u^\circ - c)\,\frac{d^2\varphi}{dy^2} + \frac{27}{2}\, i\alpha\tau^2 y + 3yi\alpha^3\tau^2\,(u^\circ - c),$$

$$\theta_1 = 1 - \tau i\alpha\,[{}^3/_2\,(1 - y^2) - c] -$$

$$- \tau^2\,[9y^2 + {}^9/_4\alpha^2\,(1 - y^2)^2 - 3c\alpha^2\,(1 - y^2) + \alpha^2 c^2],$$

$$\theta_2 = \tau - \tau^2 i\alpha\,[{}^3/_2\,(1 - y^2) - c]. \qquad (2.3)$$

This system takes into account all terms up to and including τ^2. The boundary conditions (2.2) yield the relations

$$\eta = \frac{\varphi(0)}{c - {}^3/_2}\, e^{i\alpha(x - ct)},$$

$$2i\alpha\left[\frac{d\varphi}{dy}\right]_0 + 3\,\mathrm{ctg}\,\beta\,\frac{\varphi(0)}{c - {}^3/_2} + f(0)\,R + \alpha^2 RS\,\frac{\varphi(0)}{c - {}^3/_2} = 0,$$

$$\frac{3\varphi(0)}{R\,(c - {}^3/_2)} = s_{xy}{}^*\,(0). \qquad (2.4)$$

Eliminating f(y) from (2.3), we obtain an equation for $\varphi(y)$

$$i\alpha\left[(u^\circ - c)\left(\frac{d^2\varphi}{dy^2} - \alpha^2\varphi\right) - \frac{d^2u^\circ}{dy^2}\,\varphi\right] = 2i\alpha\,\frac{ds_{xx}^*}{dy} + \frac{d^2 s_{xy}^*}{dy^2} + \alpha^2 s_{xy}^*. \quad (2.5)$$

Here s_{xx}^* and s_{xy}^* have the form given in (2.3).

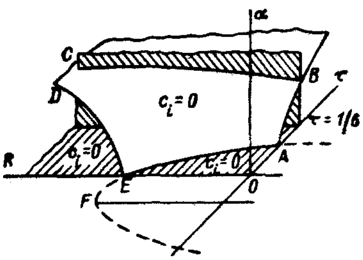

Fig. 2

Using the first equation in (2.3), we eliminate f(0) from the boundary condition (2.4). The boundary condition for equation (2.5) at y = 0 is then

$$\frac{\alpha}{c - {}^3/_2}\,(3\,\mathrm{ctg}\,\beta + \alpha^2 RS)\,\varphi(0) + \alpha\left\{2i\alpha\left[\frac{d\varphi}{dy}\right]_0 + \right.$$

$$\left. + R\left(c - \frac{3}{2}\right)\left[\frac{d\varphi}{dy}\right]_0 + Rs_{xy}^*(0)\right\} -$$

$$- iR\left[\frac{ds_{xy}^*}{dy}\right]_0 = 0, \qquad \frac{3\varphi(0)}{R\,(c - {}^3/_2)} = s_{xy}^*(0). \qquad (2.6)$$

The conditions u' (t, x, 1) = v' (t, x, 1) = 0 yield

$$\varphi(1) = \left[\frac{d\varphi}{dy}\right]_1 = 0. \qquad (2.7)$$

The problem of solving (2.5) with the boundary conditions (2.6), (2.7) leads, as is well known [4], to the problem of finding c = c (R, α, β, τ).

3. To find c = c (R, α, β, τ) for small α we expand φ and c in terms of the parameter α:

$$\varphi = \varphi^\circ + \alpha\varphi' + \alpha^2\varphi'' + \ldots, \quad c = c^\circ + \alpha c' + \alpha^2 c'' + \ldots. \quad (3.1)$$

Substitute (3.1) into (2.5)–(2.7) and (2.3). For α = 0 we obtain

$$\frac{d^2}{dy^2}\left[(1 - 27\tau^2 y^2)\,\frac{d^2\varphi^\circ}{dy^2}\right] = 0, \qquad \left[\frac{d^3\varphi^\circ}{dy^3}\right]_0 = 0,$$

$$\frac{3\varphi^\circ(0)}{c^\circ - {}^3/_2} = \left[\frac{d^2\varphi^\circ}{dy^2}\right]_0, \quad \varphi^\circ(1) = \left[\frac{d\varphi^\circ}{dy}\right]_1 = 0. \qquad (3.2)$$

From (3.2) we find [4] the solution to the zeroth approximation

$$c^\circ = 3 + {}^{81}/_4\tau^2, \qquad \varphi^\circ = (1 - y)^2 + \tau^2\,(9y - {}^{27}/_2 y^2 + {}^9/_2 y_4). \quad (3.3)$$

The variable $\varphi'(y)$ is then determined by the equation

$$\frac{d^2}{dy^2}\left[(1 - 27\tau^2 y^2)\,\frac{d^2\varphi'}{dy^2}\right] = iR\left[(u^\circ - c^\circ)\,\frac{d^2\varphi^\circ}{dy^2} - \frac{d^2 u_0}{dy^2}\,\varphi^\circ\right]. \quad (3.4)$$

with the boundary conditions

$$\frac{1}{c^\circ - {}^3/_2}\,[3\,\mathrm{ctg}\,\beta + \alpha^2 RS]\,\varphi^\circ(0) +$$

$$+ R\left(c^\circ - \frac{3}{2}\right)\left[\frac{d\varphi'}{dy}\right]_0 - i\left[\frac{d^3\varphi'}{dy^3}\right]_0 + 3\tau\left[\frac{d\varphi^\circ}{dy}\right]_0 = 0,$$

$$- \frac{3\varphi'(0)}{c^\circ - {}^3/_2} + \frac{3\varphi^\circ(0)\,c'}{(c^\circ - {}^3/_2)^2} + \left[\frac{d^2\varphi'}{dy^2}\right]_0 = 0, \quad \varphi'(1) = \left[\frac{d\varphi'}{dy}\right]_1 = 0. \quad (3.5)$$

Integrating (3.4), taking into account (3.3) and the boundary conditions (3.5), we find

$$c' = i\,(1 + {}^{27}/_4\tau^2)\,[{}^6/_5 R\,(1 + 9.12\tau^2) -$$

$$- (1 - 3.30\tau^2)\,\mathrm{ctg}\,\beta + 3\tau - {}^1/_3 RS\,(1 - 3.30\tau^2)]. \qquad (3.6)$$

Here, as everywhere else, we neglect terms of orders higher than τ^2.

For small α we retain the two leading terms in the expansions. We obtain then c = c° + αc', where c° and c' are given by (3.3) and (3.6). Equating the imaginary part of c with zero, we find

$$\alpha\,[{}^6/_5 R\;\;(1 + 9.12\tau^2) - (1 - 3.30\tau^2)\;\;\mathrm{ctg}\,\beta +$$

$$3\tau - {}^1/_3\alpha^2 RS\;\;(1 - 3.30\tau^2)] = 0. \qquad (3.7)$$

For β = const, Eq. (3.7) defines a neutral surface in the R, α, M space. This surface consists of the plane α = 0 and the surface ABCDE, which intersect along the parabola (Fig. 2)

$$R = {}^5/_6\,\mathrm{ctg}\,\beta\,(1 - 10.35\tau^2) - 2.50\tau. \qquad (3.8)$$

The portion EA of the parabola (3.8) is a branching line of the neutral surface.

For $\tau = \lambda = 0$ we have in the α, R plane a neutral curve, consisting of the axis α = 0 and the line ED. The branch point E defines then [4] the minimal critical value R = 5/6 ctg β for S ≠ 0.

From (3.7) one can see that when S ≠ 0 the inequality

$$R \leqslant {}^5/_6\,(1 - 10.35\tau^2)\,\mathrm{ctg}\;\beta - 2.50\tau$$

must hold on the surface ABCDE if α is to be real. Therefore for small α the surface ABCDE cannot intersect the α axis.

Taking account of (3.6) one can easily see that when $\tau < 1/6$, S ≠ 0, the values of c_i are negative for $\alpha > \alpha_0$ (α_0 are the values of α on the surface ABCDE) and positive for $\alpha < \alpha_0$.

Taking into account that $\lambda = \mu/G$, where G is the elastic shear modulus, we find from (1.3) that τ is independent of μ. Therefore, if τ = 0 because of G = ∞, then for R = 0, μ = ∞ we find [4] that the surface ABCDE does not intersect the α axis in the case S ≠ 0. For S = 0 the surface ABCDE intersects the α axis at α = ∞.

For a Newtonian fluid (G = ∞) the branch point E of the neutral curve coincides with the origin for $\beta = \pi/2$. For a viscoelastic fluid with G = const ≠ ∞, the Reynolds number is equal to zero at the branch point of the neutral curve when

$$\tau = A = 1/17.25\,[-2.50 + (6.25 + 2.88\,\mathrm{ctg}^2\,\beta)^{1/2}]\,\mathrm{tg}\,\beta.$$

Consequently, in the case of a Newtonian fluid the critical Reynolds number is equal to zero for flow on a vertical plane ($\beta = \pi/2$), whereas in the case of a viscoelastic fluid the critical Reynolds number is equal to zero for β = arc ctg (3τ). Thus, for example, for τ = 0.1 the critical Reynolds number is equal to zero for flow along a plane inclined at $\beta \approx 73°$ with respect to the horizontal.

Fig. 2 shows the neutral surface for the case S ≠ 0. The equations of motion (1.3) have an exact solution, corresponding to rectilinear laminar flow, for $\tau < 1/6$, and this flow is stable with respect to two-dimensional perturbations in the region between the planes $\alpha = 0$, $\tau = 0$, $R = 0$ and the surface ABCDE. The plane $\alpha = 0$ and the surface ABCDE are a neutral surface, whose branching line EA is an arc of the parabola (3.8) in the plane $\alpha = 0$. The parabola has an axis parallel to the R axis, and its vertex is at the point F, whose coordinates are

$$\alpha = 0, \quad R = 0.83 \operatorname{ctg} \beta + 0.18 \operatorname{tg} \beta, \quad \tau = -0.15 \operatorname{tg} \beta.$$

The flow of a film of viscoelastic fluid is less stable than the flow of a Newtonian fluid with the same viscosity.

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